Available online at www.sciencedirect.com
ScienceDirect
Advances in Mathematics 216 (2007) 279–301
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

A new theory of regular functions of a quaternionic variable

Graziano Gentili [a,*], Daniele C. Struppa [b]

[a] Dipartimento di Matematica "U. Dini", Università di Firenze, Viale Morgagni, 67 A, Firenze, Italy
[b] Department of Mathematics and Computer Science, Chapman University, Orange, CA 92866, USA

Received 24 April 2006; accepted 22 May 2007

Available online 2 June 2007

Communicated by Michael J. Hopkins

Abstract

In this paper we develop the fundamental elements and results of a new theory of regular functions of one quaternionic variable. The theory we describe follows a classical idea of Cullen, but we use a more geometric formulation to show that it is possible to build a rather complete theory. Our theory allows us to extend some important results for polynomials in the quaternionic variable to the case of power series.

Keywords: Functions of hypercomplex variables; Functions of complex variables; Quaternions

1. Introduction

Let $\mathbb{H}$ denote the skew field of real quaternions. Its elements are of the form $q = x_0 + ix_1 + jx_2 + kx_3$ where the x_l are real, and i, j, k, are imaginary units (i.e. their square equals -1) such that $ij = -ji = k$, $jk = -kj = i$, and $ki = -ik = j$. Since the beginning of the last century, mathematicians have been interested in creating a theory of quaternionic valued functions of a quaternionic variable, which would somehow resemble the classical theory of holomorphic functions of one complex variable.

* Corresponding author.
E-mail addresses: gentili@math.unifi.it (G. Gentili), struppa@chapman.edu (D.C. Struppa).

0001-8708/$ – see front matter
doi:10.1016/j.aim.2007.05.010

Up to now several interesting theories have been introduced. The best known is the one due to Fueter [5], who defined the differential operator

$$\frac{\partial}{\partial \bar{q}} = \frac{1}{4}\left(\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}\right)$$

now known as the Cauchy–Fueter operator and defined the space of regular functions as the space of solutions of the equation associated to this operator. This theory of regular functions is by now very well developed, in many different directions, and we refer the reader to [13] for the basic features of these functions. More recent work in this area includes [7], and references therein. While the theory is extremely successful in replicating many important properties of holomorphic functions (and not only in one variable, see [2]), the major disappointment is that even the identity $f(q) = q$, and therefore polynomials and series, fail to be regular in the sense of Fueter.

A second, not as well known, definition was given by Cullen in [3] on the basis of the notion of intrinsic functions as developed in [12]. This definition has the advantage that polynomials and even power series of the form $\sum_{n=0}^{\infty} q^n a_n$ are regular in this sense.

Polynomials of a quaternionic variable, and power series, are also introduced in the interesting class of holomorphic functions over quaternions, which was defined by Fueter [4] and more recently generalized and developed by Laville and Ramadanoff [8,9], who built the theory of holomorphic Cliffordian functions. If Δ denotes the Laplacian, then the (left) holomorphic functions over quaternions are the solutions of the equation associated to the differential operator $\frac{\partial}{\partial \bar{q}}\Delta$. It turns out that the set of Cullen regular functions and the set of Fueter regular functions, strictly contained in the set of holomorphic functions over quaternions, do not coincide.

Cullen regular functions are also closely related to a class of functions of the reduced quaternionic variable $x_0 + ix_1 + jx_2$, studied by Leutwiler [10]. This class consists of all the solutions of a generalized Cauchy–Riemann system of equations, it contains the natural polynomials, and supports the series expansion of its elements as well.

In order to offer our generalization of Cullen's definition, [3], let us denote by $\mathbb{S}$ the unit sphere of purely imaginary quaternions, i.e. $\mathbb{S} = \{q = ix_1 + jx_2 + kx_3$ such that $x_1^2 + x_2^2 + x_3^2 = 1\}$. Notice that if $I \in \mathbb{S}$, then $I^2 = -1$; for this reason the elements of $\mathbb{S}$ are called imaginary units. The following proposition, whose proof is straightforward, is used to give the definition of regularity.

Proposition 1.1. *For any non-real quaternion $q \in \mathbb{H}\setminus\mathbb{R}$, there exist, and are unique, $x, y \in \mathbb{R}$ with $y > 0$, and $I \in \mathbb{S}$ such that $q = x + yI$.*

Definition 1.2. Let Ω be a domain in $\mathbb{H}$. A real differentiable function $f : \Omega \to \mathbb{H}$ is said to be C-regular if, for every $I \in \mathbb{S}$, its restriction f_I to the complex line $L_I = \mathbb{R} + \mathbb{R}I$ passing through the origin and containing 1 and I is holomorphic on $\Omega \cap L_I$.

Throughout the paper, since no confusion can arise, we will refer to C-regular functions as regular functions tout court.

Remark 1.3. The requirement that $f : \Omega \to \mathbb{H}$ is regular is equivalent to require that, for every I in $\mathbb{S}$,

$$\bar{\partial}_I f(x+yI) := \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x+yI) = 0,$$

on $\Omega \cap L_I$.

Remark 1.4. The Cauchy–Fueter differential operator defines a derivative in the classical Fréchet sense; on the other hand, the definition of regularity which we have just provided can be interpreted in the spirit of the Gateaux derivative.

Still in the spirit of Gateaux, we can define a notion of I-derivative as follows:

Definition 1.5. Let Ω be a domain in $\mathbb{H}$ and let $f : \Omega \to \mathbb{H}$ be a real differentiable function. For any $I \in \mathbb{S}$ and any point $q = x + yI$ in Ω (x and y are real numbers here) we define the I-derivative of f in q by

$$\partial_I f(x+yI) := \frac{1}{2}\left(\frac{\partial}{\partial x} - I\frac{\partial}{\partial y}\right) f_I(x+yI).$$

In this paper we prove several results which show that, on the basis of these definitions, it is possible to construct a significant and interesting theory for regular functions. Our first results deal with issues of convergence of power series, the identity principle, the maximum modulus principle, the Cauchy representation formula, the Liouville theorem and the Morera theorem. We then prove a version of the Schwarz lemma and we are also able to make some advances in the study of the geometry of the unit ball, of the four dimensional analog of the Siegel right-half plane (biregular to the unit ball via the analogous of a Cayley), and their transformations. These results were announced in [6]. Finally, the last section of this paper describes the zeroes of regular functions, with a result which extends [11]. We plan to come back to these and other related issues in future papers.

2. Power series and series expansions for regular functions

In order to study polynomials and power series in q, we first note that the basic polynomial $q^n a$, with a a quaternion, is regular according to Definition 1.2. Since the sum of regular functions is regular, we immediately have that polynomials with quaternionic coefficients on the right are regular. In order to consider power series $\sum_{n=0}^{\infty} q^n a_n$, we will endow the space of regular functions with the natural uniform convergence on compact sets. The same arguments which hold for complex power series, see e.g. [1], allow to obtain the analog of the Abel's theorem.

Theorem 2.1. *For every power series $\sum_{n=0}^{\infty} q^n a_n$ there exists a number R, $0 \leqslant R \leqslant \infty$, called the radius of convergence, such that the series converges absolutely for every q with $|q| < R$ and uniformly for every q with $|q| \leqslant \rho < R$. Moreover if $|q| > R$, the series is divergent.*

Since convergence of power series is uniform on compact sets, it turns out that power series are regular in their domain of convergence. Note also that every power series is also real analytic.

The first important consequence of our definition of regularity is that, for regular functions, we can introduce a notion of derivative.

Definition 2.2. Let Ω be a domain in $\mathbb{H}$, and let $f : \Omega \to \mathbb{H}$ be a regular function. The Cullen derivative of f, $\partial_C f$, is defined as follows:

$$\partial_C(f)(q) = \begin{cases} \partial_I(f)(q) & \text{if } q = x + yI \text{ with } y \neq 0, \\ \frac{\partial f}{\partial x}(x) & \text{if } q = x \text{ is real.} \end{cases}$$

This definition of derivative is well posed because it is applied only to regular functions. In fact, the value of the derivative at a real point x can be computed using different imaginary units, and a priori there is no reason why the values which one obtains should coincide. However, if a function f is regular, its derivative in the point x is immediately shown to be equal to $\frac{\partial f}{\partial x}(x)$. It is easy to construct examples which manifest this problem if f is not regular. Note that this phenomenon is peculiar of the quaternionic case, and does not appear in the complex case. The reason for this is that the unit sphere of imaginary numbers has dimension 2 in the case of quaternions, but it is only made of two points, $\{i, -i\}$, in the complex case.

Let f be a regular function. Since for every I in $\mathbb{S}$ it is $\bar{\partial}_I(\partial_C(f)) = \partial_C(\bar{\partial}_I(f)) = 0$ we obtain that the Cullen derivative of a regular function is still regular.

Note also that the derivative of a power series can be done term by term because of the uniform convergence, so that

$$\partial_C\left(\sum_{n=0}^{\infty} q^n a_n\right) = \sum_{n=1}^{\infty} q^{n-1} n a_n.$$

This new series has the same radius of convergence of the original series.

In what follows, we will always restrict our attention to functions which are regular on a ball $B(0, R)$ centered in the origin and of radius R.

In order for us to study regular functions, we will need a simple representation of the restriction of a regular function as a pair of holomorphic functions. To do so, we need a few simple preliminary results on the set $\mathbb{S}$.

Proposition 2.3. *Let $I = iI_1 + jI_2 + kI_3$ and $J = iJ_1 + jJ_2 + kJ_3$ be two elements in $\mathbb{S}$, let $\langle I, J\rangle = I_1J_1 + I_2J_2 + I_3J_3 \in \mathbb{R}$ denote the Euclidean scalar product of their coordinates, and let $I \times J = i(I_2J_3 - I_3J_2) + j(I_3J_1 - I_1J_3) + k(I_1J_2 - I_2J_1) \in \mathbb{R} \cdot \mathbb{S}$ be their natural vector product. Then the quaternionic product IJ can be computed through the following formula*:

$$IJ = -\langle I, J\rangle + I \times J.$$

Proof. The result follows immediately from the direct computation of the product $IJ = (iI_1 + jI_2 + kI_3)(iJ_1 + jJ_2 + kJ_3)$. $\square$

Note that the previous computation shows, in particular, that the product of two orthogonal elements of $\mathbb{S}$ lies in $\mathbb{S}$ as well. We will use this simple fact to build orthogonal bases in $\mathbb{S}$.

Proposition 2.4. *Let I and J be two orthogonal elements in $\mathbb{S}$, and let $K = IJ$. Then*:

1. *$K = IJ = -JI$ is an element of $\mathbb{S}$,*
2. *K is orthogonal to both I and J,*
3. *$JK = I = -KJ$ and $KI = J = -IK$.*

Proof. We will prove the three statements independently.

1. This follows immediately from the previous proposition, noting that I and J are orthogonal, and that clearly $I \times J = -J \times I$.
2. This again is a consequence of the previous proposition, of the orthogonality of I and J, and of the fact that $I \times J$ is always orthogonal to both I and J. These three facts imply that

$$\langle K, I\rangle = \langle IJ, I\rangle = \langle I \times J, I\rangle = 0.$$

3. Here we apply repeatedly the previous proposition to obtain the sequence of equalities

$$\begin{aligned} JK &= J(IJ) = J\bigl(-\langle I, J\rangle + I \times J\bigr) \\ &= -\bigl\langle J, -\langle I, J\rangle + I \times J\bigr\rangle + J \times \bigl(-\langle I, J\rangle + I \times J\bigr). \end{aligned}$$

Using the orthogonality of I and J we obtain

$$JK = -\langle J, I \times J\rangle + J \times (I \times J).$$

Now note that $\langle J, I \times J\rangle = 0$ because $I \times J$ is orthogonal to J, and that, by the same reason, $IJ = I \times J$. Thus to conclude the proof we only need to show that $J \times K = I$. This can be obtained by a direct computation, which we leave to the reader, and which uses once again the orthogonality of I and J. □

The result we have just proved is simple, but it shows that we can use I, J, and K as a basis for $\mathbb{S}$; moreover, given any element I in $\mathbb{S}$, we can always construct such a basis (though not in a unique way, as the basis will ultimately depend on the choice of J among vectors which are orthogonal to I).

The following lemma (we will often refer to it as the splitting lemma) is simple to prove but is essential for all the results in this paper.

Lemma 2.5. *If f is a regular function on $B = B(0, R)$, then for every $I \in \mathbb{S}$, and every J in $\mathbb{S}$, perpendicular to I, there are two holomorphic functions $F, G : B \cap L_I \to L_I$ such that for any $z = x + yI$, it is*

$$f_I(z) = F(z) + G(z)J.$$

Proof. Given any pair of orthogonal vectors I and J in $\mathbb{S}$, consider the third element K of the orthogonal basis I, J, K, and write $f_I(x + yI) = f(x + yI)$ as $f = f_0 + If_1 + Jf_2 + Kf_3$. Since f is regular, we know that $(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y})f_I(x + yI) = 0$, i.e.

$$\frac{\partial f_0}{\partial x} + I\frac{\partial f_1}{\partial x} + J\frac{\partial f_2}{\partial x} + K\frac{\partial f_3}{\partial x} + I\left(\frac{\partial f_0}{\partial y} + I\frac{\partial f_1}{\partial y} + J\frac{\partial f_2}{\partial y} + K\frac{\partial f_3}{\partial y}\right) = 0.$$

The expression above can be rewritten (taking advantage of the properties of the imaginary units) as

$$\frac{\partial f_0}{\partial x} - \frac{\partial f_1}{\partial y} + I\left(\frac{\partial f_0}{\partial y} + \frac{\partial f_1}{\partial x}\right) + J\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_3}{\partial y}\right) + K\left(\frac{\partial f_3}{\partial x} + \frac{\partial f_2}{\partial y}\right) = 0.$$

This implies that the functions $f_0 + If_1$ and $f_2 + If_3$ satisfy the standard Cauchy–Riemann system and therefore they are both holomorphic. In particular, if we set $f_0 + If_1 = F$, and $f_2 + If_3 = G$, we obtain that $f_I(x + yI) = F(x + yI) + G(x + yI)J$, and so the lemma is demonstrated once we set $z = x + yI$. $\square$

Given that the functions F and G are holomorphic on the plane $\mathbb{R} + \mathbb{R}I$, it is not surprising (we will show it in a second) that f admits, on that plane, a series expansion in powers of z. What is more surprising is the fact that such an expansion can be used to provide a series expansion for f in powers of q. This is a crucial result for this theory, and its proof requires one more preliminary step.

Proposition 2.6. *Let $f : B \to \mathbb{H}$ be a regular function. Then, for any $n \in \mathbb{N}$, its Cullen derivative $\partial_C^n f : B \to \mathbb{H}$ is regular and it is $\partial_C^n f(x + yI) = \frac{\partial^n f}{\partial x^n}(x + yI)$.*

Proof. The fact that $\partial_C^n f$ is well defined has already been established. To prove the equality $\partial_C^n f(x + yI) = \frac{\partial^n f}{\partial x^n}(x + yI)$ we proceed by induction. First we note that the equality is trivial for $n = 1$, since

$$\partial_C f(x + yI) = \frac{1}{2}\left(\frac{\partial}{\partial x} - I\frac{\partial}{\partial y}\right)f(x + yI) = \frac{1}{2}\left(\frac{\partial f}{\partial x} - I\frac{\partial f}{\partial y}\right)(x + yI) = \frac{\partial f}{\partial x}(x + yI).$$

To prove the induction step note that since f is regular, then

$$\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right)\left(\frac{\partial^n f}{\partial x^n}\right) = \frac{\partial^{n+1} f}{\partial x^{n+1}} + I\frac{\partial^{n+1} f}{\partial x^n \partial y} = \frac{\partial^n}{\partial x^n}\left(\frac{\partial f}{\partial x} + I\frac{\partial f}{\partial y}\right) = 0.$$

Thus we have that

$$\frac{\partial^{n+1} f}{\partial x^{n+1}} = -I\frac{\partial^{n+1} f}{\partial x^n \partial y}$$

and therefore (by the induction hypothesis)

$$\begin{aligned}\partial_C^{n+1} f = \partial_C\left(\partial_C^n f\right) = \partial_C\left(\frac{\partial^n f}{\partial x^n}\right) &= \frac{1}{2}\left(\frac{\partial}{\partial x} - I\frac{\partial}{\partial y}\right)\left(\frac{\partial^n f}{\partial x^n}\right)\\ &= \frac{1}{2}\left(\frac{\partial^{n+1} f}{\partial x^{n+1}} - I\frac{\partial^{n+1} f}{\partial x^n \partial y}\right) = \frac{\partial^{n+1} f}{\partial x^{n+1}}.\end{aligned}$$

This concludes the proof. $\square$

It is now possible to deduce the following important result.

Theorem 2.7. *A function $f : B \to \mathbb{H}$ is regular if, and only if, it has a series expansion of the form*

$$f(q) = \sum_{n=0}^{\infty} q^n \frac{1}{n!} \frac{\partial^n f}{\partial x^n}(0)$$

converging on B. In particular if f is regular then it is C^∞ on B.

Proof. Consider, in the complex plane L_I, the disc Δ_I centered in the origin and with radius $a > 0$, where $a < R$. Then we can use the representation from the splitting Lemma 2.5 to find an integral representation for f_I inside Δ_I. Specifically, using the fact that both F and G are holomorphic in the domain $B \cap L_I$ of the complex plane L_I, with values in the same complex plane L_I, we obtain $(\zeta - z)^{-1} F(z) = F(z)(\zeta - z)^{-1}$ and $(\zeta - z)^{-1} G(z) = G(z)(\zeta - z)^{-1}$, for any $\zeta \neq z \in B \cap L_I$. Therefore for any z in Δ_I we have:

$$f_I(z) = \frac{1}{2\pi I} \int_{\partial \Delta_I} \frac{F(\zeta)}{\zeta - z} \, d\zeta + \left(\frac{1}{2\pi I} \int_{\partial \Delta_I} \frac{G(\zeta)}{\zeta - z} \, d\zeta \right) J.$$

Each of these two integrals may now be transformed into a power series as in classical complex analysis. For example (and the same process can be applied to the integral containing G) one has, for any $z \in \Delta_I$,

$$\int_{\partial \Delta_I} \frac{1}{1 - \frac{z}{\zeta}} \frac{F(\zeta)}{\zeta} \, d\zeta = \int_{\partial \Delta_I} \sum_{n \geqslant 0} \left(\frac{z}{\zeta} \right)^n \frac{F(\zeta)}{\zeta} \, d\zeta = \sum_{n \geqslant 0} z^n \left(\int_{\partial \Delta_I} \frac{F(\zeta)}{\zeta^{n+1}} \, d\zeta \right). \tag{1}$$

Notice that in the above formula we have chosen to put $(\frac{z}{\zeta})^n$ on the left (instead of on the right) of $\frac{F(z)}{\zeta}$ so that the power series will have its coefficients on the right, and will be regular in its domain of convergence. Equality (1) immediately yields that

$$\begin{aligned} f_I(z) &= \sum_{n \geqslant 0} z^n \frac{1}{n!} \frac{\partial^n F}{\partial z^n}(0) + \sum_{n \geqslant 0} z^n \frac{1}{n!} \frac{\partial^n G}{\partial z^n}(0) J = \sum_{n \geqslant 0} z^n \frac{1}{n!} \left(\frac{\partial^n (F + GJ)}{\partial z^n}(0) \right) \\ &= \sum_{n \geqslant 0} z^n \frac{1}{n!} \left(\frac{\partial^n f}{\partial z^n}(0) \right). \end{aligned}$$

Now, because of the last proposition, we can transform this equation as follows:

$$f_I(z) = \sum_{n \geqslant 0} z^n \frac{1}{n!} \left(\frac{\partial^n f}{\partial z^n}(0) \right) = \sum_{n \geqslant 0} z^n \frac{1}{n!} \left(\frac{1}{2} \left(\frac{\partial}{\partial x} - I \frac{\partial}{\partial y} \right) \right)^n f(0) = \sum_{n \geqslant 0} z^n \frac{1}{n!} \frac{\partial^n f}{\partial x^n}(0).$$

In particular this shows that $f_I(z)$ can be given a series representation in z^n with coefficients $a_n = \frac{1}{n!} \frac{\partial^n f}{\partial x^n}(0)$ which do not depend at all on the choice of I. Therefore the representation we have found holds for any $I \in \mathbb{S}$, and this concludes the proof. $\square$

Corollary 2.8. *Let $f : B \to \mathbb{H}$ be regular. If there exists $I \in \mathbb{S}$ such that $f(L_I) \subseteq L_I$, then the series expansion of f*

$$f(q) = \sum_{n=0}^{\infty} q^n \frac{1}{n!} \frac{\partial^n f}{\partial x^n}(0)$$

has all its coefficients in L_I.

Proof. If $I \in \mathbb{S}$ is such that $f(L_I) \subseteq L_I$, then for any real number x we have $f(x) = f_I(x) \in L_I$. Therefore $\frac{\partial^n f}{\partial x^n}(x) \in L_I$ for any $n \in \mathbb{N}$, $x \in \mathbb{R}$, and the conclusion follows. □

3. Cauchy integral formulas

The power series expansion which we have proved for regular functions in the last section is the key ingredient in proving the analog, for regular functions, of many well-known results from the theory of holomorphic functions in one variable, such as the identity principle, the maximum modulus, the Cauchy representation and estimates, and the Liouville and Morera theorems. This short section is dedicated to the proofs of such results.

We begin with the proof of a version of the identity principle.

Theorem 3.1. *Let $f : B \to \mathbb{H}$ be a regular function. Denote by $Z_f = \{q \in B\colon\ f(q) = 0\}$ the zero set of f. If there exists $I \in \mathbb{S}$ such that $L_I \cap Z_f$ has an accumulation point, then $f \equiv 0$ on B.*

Proof. On $L_I \cap B$ we can write

$$f(x + yI) = F(x + yI) + G(x + yI)J$$

with F and G holomorphic functions on L_I. Now, under the assumption that $L_I \cap Z_f$ has an accumulation point, we deduce that both F and G are identically zero on $L_I \cap B$. This implies, in particular, that $\frac{\partial^n f}{\partial x^n}(0) = 0$ for all values of n. Since these derivatives are the coefficients of the power series expansion of f, this implies that $f \equiv 0$ on B. □

This result immediately implies the following corollary.

Corollary 3.2. *Let f and g be regular functions on the ball B. If there exists $I \in \mathbb{S}$ such that $f \equiv g$ on a subset of $L_I \cap B$ having an accumulation point in $L_I \cap B$, then $f \equiv g$ everywhere on B.*

Before we can prove our next objective, the maximum principle, we need a preliminary result on the mean value property.

Proposition 3.3. *If $f : B \to \mathbb{H}$ is a regular function, and if $I \in \mathbb{S}$, then $f_I : L_I \cap B \to \mathbb{H}$ has the mean value property.*

Proof. We know, from Lemma 2.5, that we can write $f_I(x + yI) = F(x + yI) + G(x + yI)J$. Therefore, for all points a in $L_I \cap B$, and all positive numbers r such that $\overline{\Delta(a; r)} \subset L_I \cap B$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} f_I\big(a+re^{I\theta}\big)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\big(F\big(a+re^{I\theta}\big)+G\big(a+re^{I\theta}\big)J\big)\,d\theta = F(a)+G(a)J = f_I(a).$$

This concludes the proof. □

We are now in a position to prove the maximum modulus principle for regular functions.

Theorem 3.4. *Let $f : B \to \mathbb{H}$ be a regular function. If $|f|$ has a relative maximum at a point $a \in B$, then f is constant on B.*

Proof. If $f(a)=0$ the result is trivial. We will assume therefore that $f(a)\neq 0$. By multiplying f, if necessary, by a quaternion, we can reduce the theorem to the case in which $f(a)>0$. Let then I be the normalized imaginary part of $a = x_0 + y_0 I$, so that I is an element of $\mathbb{S}$, and consider the function f_I. As customary we set, for $r>0$ sufficiently small,

$$M(r)=\sup_{\theta\in\mathbb{R}}\big\{\big|f\big(a+re^{I\theta}\big)\big|\big\}.$$

By hypothesis, we have that $f(a)\geqslant M(r)$ when r is sufficiently small. On the other hand, since f_I satisfies the mean value property from the previous proposition, we immediately obtain that $f_I(a)=M(r)$. Set now $z=x+yI$ so that, for sufficiently small $r=|z-a|$, the function $g(z)=\mathrm{Re}(f_I(a)-f_I(z))$ is non-negative. In fact, we have that $g(z)=0$ if and only if $f_I(z)=f_I(a)$. By the mean value property

$$f_I(a)=\frac{1}{2\pi}\int_0^{2\pi} f_I\big(a+re^{I\theta}\big)\,d\theta$$

and by taking into account that the real part of a holomorphic map also satisfies the mean value property, we obtain

$$g(a)=\frac{1}{2\pi}\int_0^{2\pi} g\big(a+re^{I\theta}\big)\,d\theta = 0.$$

At the same time, we know that g is continuous and non-negative on $\partial\Delta(a,r)$, and so we obtain that $g(a+re^{I\theta})=0$ for all θ in $\mathbb{R}$. As a consequence, $g(z)$ is identically zero in the closed disc, and therefore $f_I(z)=f_I(a)$ for all points z in $\overline{\Delta(a,r)}$. Since this last set clearly has an accumulation point in $L_I\cap B$, we use the identity principle to conclude the proof. □

Maybe the most important consequence of Lemma 2.5 is the analog, for regular functions, of the Cauchy representation formula. In order to state it appropriately, we will adopt the following notation. If $q\in B$, we set

$$I_q=\begin{cases}\frac{\mathrm{Im}(q)}{|\mathrm{Im}(q)|}\in\mathbb{S} & \text{if } \mathrm{Im}(q)\neq 0,\\ \text{any element of } \mathbb{S} & \text{otherwise.}\end{cases}$$

Notice that for any $\zeta \in L_{I_q}$, $\zeta \neq q$ the equality $(\zeta - q)^{-1} d\zeta = d\zeta(\zeta - q)^{-1}$ holds. We can now prove the integral representation formula.

Theorem 3.5. *Let $f : B \to \mathbb{H}$ be a regular function, and let $q \in B$. Then*

$$f(q) = \frac{1}{2\pi I_q} \int\limits_{\partial\Delta_q(0,r)} \frac{d\zeta}{(\zeta - q)} f(\zeta)$$

where $\zeta \in L_{I_q} \cap B$, and where $r > 0$ is such that

$$\overline{\Delta_q(0,r)} = \left\{x + yI_q\colon\ x^2 + y^2 \leqslant r^2\right\}$$

is contained in B and contains q.

Proof. The result follows immediately from the splitting lemma, as indicated by the following equalities.

$$\begin{aligned}
\frac{1}{2\pi I_q} \int\limits_{\partial\Delta_q(0,r)} \frac{d\zeta}{\zeta - q} f(\zeta) &= \frac{1}{2\pi I_q} \int\limits_{\partial\Delta_q(0,r)} \frac{d\zeta}{\zeta - q} f_{I_q}(\zeta) \\
&= \frac{1}{2\pi I_q} \int\limits_{\partial\Delta_q(0,r)} \frac{d\zeta}{\zeta - q} \big(F(\zeta) + G(\zeta)J\big) \\
&= \frac{1}{2\pi I_q} \int\limits_{\partial\Delta_q(0,r)} \frac{F(\zeta)}{\zeta - q} d\zeta + \left(\frac{1}{2\pi I_q} \int\limits_{\partial\Delta_q(0,r)} \frac{G(\zeta)}{\zeta - q} d\zeta\right) J \\
&= F(q) + G(q)J = f(q). \qquad \square
\end{aligned} \tag{2}$$

As a consequence we obtain:

Theorem 3.6 *(Cauchy estimates). Let $f : B(0,R) \to \mathbb{H}$ be a regular function, let $r < R$, $I \in \mathbb{S}$, and $\partial\Delta_I(0,r) = \{(x + yI)\colon\ x^2 + y^2 = r^2\}$. If $M_I = \max\{|f(q)|\colon\ q \in \partial\Delta_I(0,r)\}$ and if $M = \inf\{M_I\colon\ I \in \mathbb{S}\}$, then*

$$\frac{1}{n!}\left|\frac{\partial^n f}{\partial x^n}(0)\right| \leqslant \frac{M}{r^n}, \quad n \geqslant 0.$$

Proof. The result follows the same ideas as in the case of holomorphic functions of a complex variable. Specifically, the proof of the series representation for a regular function shows that, for any $I \in \mathbb{S}$, we can write

$$\frac{1}{n!}\frac{\partial^n f}{\partial x^n}(0) = \frac{1}{2\pi I} \int\limits_{\partial\Delta_I(0,r)} \frac{d\zeta}{\zeta^{n+1}} f(\zeta).$$

Therefore, for any $I \in \mathbb{S}$ we can write the following sequence of inequalities:

$$\frac{1}{n!}\left|\frac{\partial^n f}{\partial x^n}(0)\right| \leqslant \frac{1}{2\pi} \int\limits_{\partial \Delta_I(0,r)} \frac{|f(\zeta)|}{r^{n+1}}\,d\zeta \leqslant \frac{1}{2\pi} \int\limits_{\partial \Delta_I(0,r)} \frac{M_I}{r^{n+1}}\,d\zeta = \frac{M_I}{r^n}.$$

By taking the infimum, for $I \in \mathbb{S}$, of the right-hand side of the inequality we prove the assertion. □

We now have all the instruments needed to prove the analog of the Liouville theorem.

Theorem 3.7. *Let $f : \mathbb{H} \to \mathbb{H}$ be an entire regular map (i.e. a regular map defined and regular everywhere on $\mathbb{H}$). If f is bounded, i.e. there exists a positive number M such that $|f(q)| \leqslant M$ on all of $\mathbb{H}$, then f is constant.*

Proof. The Cauchy estimates yield that, for any $r \in \mathbb{R}$,

$$\frac{1}{n!}\left|\frac{\partial^n f}{\partial x^n}(0)\right| \leqslant \frac{M}{r^n}.$$

By letting r go to infinity, we obtain that $\frac{\partial^n f}{\partial x^n}(0) = 0$ for any positive n, and this implies that $f \equiv f(0)$. Indeed, all the coefficients of the power series representing f must be zero, with the possible exception of the first one. □

We close this section with a version of Morera's theorem.

Theorem 3.8. *Let $f : B \to \mathbb{H}$ be a differentiable function. If, for every $I \in \mathbb{S}$, the differential form $f(z)\,dz$, $z = x + yI$, $x, y \in \mathbb{R}$, defined on $L_I \cap B$ is closed, then the function f is regular.*

Proof. The hypotheses imply, by the classical Morera theorem, that each function $f_I : L_I \cap B \to \mathbb{H}$ is holomorphic. This concludes the proof, in view of our definition of regularity. □

4. The Schwarz lemma and the geometry of the unit ball

In this section we set the initial stage for the study of the geometry of the unit ball B in $\mathbb{H}$ using regular functions. We believe that this field is quite open and that one can obtain much deeper results. We plan to return to these ideas in a subsequent paper.

We begin our treatment by proving an analog of the classical Schwarz lemma. In this section B will denote the unit ball of $\mathbb{H}$, i.e. $B = \{q \in \mathbb{H}\colon |q| < 1\}$.

Theorem 4.1. *Let $f : B \to B$, $f(q) = \sum q^n a_n$, be a regular function such that $f(0) = 0$. Then, for every $q \in B$,*

$$\left|f(q)\right| \leqslant |q|$$

and

$$\left|\partial_C f(0)\right| \leqslant 1.$$

Moreover, equality holds in formulas above, at a point $q \neq 0$*, if and only if* $f(q) = qu$ *for some* $u \in \mathbb{H}$*,* $|u| = 1$*.*

Proof. Since $f(0) = 0$, we know that $a_0 = 0$ and thus $f(q) = \sum_{n\geqslant 1} q^n a_n$. Therefore the function

$$g(q) = q^{-1} f(q) = \sum_{n\geqslant 0} q^n a_{n+1}$$

is a regular map $g : B \to \mathbb{H}$, since the radius of convergence of the series representing f and g is the same. Choose now $q \in B$, $|q| < r < 1$. Then, by the maximum principle, we have

$$\left|g(q)\right| \leqslant \sup_{|w|=r} \left|g(w)\right| = \sup_{|w|=r} \frac{|f(w)|}{|w|} \leqslant \frac{1}{r}.$$

If we now let $r \to 1$, we get that $|g(q)| \leqslant 1$ on B, and since $\partial_C f(0) = g(0)$, we obtain that $|\partial_C f(0)| \leqslant 1$. If we now assume that, for some $q \in B$, it is $|f(q)| = |q|$, then, for such a value of q, we have

$$\frac{|f(q)|}{|q|} = \left|g(q)\right| = 1$$

and now by the maximum principle we obtain that $g(q) = u$ for all $q \in B$, and for a suitable $u \in \mathbb{H}$, with $|u| = 1$. Therefore we conclude that $q^{-1} f(q) = u$, and thus $f(q) = qu$. Similarly, if $|\partial_C f(0)| = 1$, we obtain that $|g(0)| = 1$ and again the thesis follows. □

Note that rational functions in the variable q are not, in general, regular functions. In particular

Proposition 4.2. *If we fix* $q_0 \in \mathbb{H} \setminus \mathbb{R}$*,* $|q_0| < 1$*, then the natural map*

$$\gamma(q) = (q - q_0)(1 - \overline{q_0} q)^{-1}$$

is not regular on B*.*

Proof. Since $q \in B$ implies $|\overline{q_0} q| < 1$, the map γ can be expanded in power series as

$$\gamma(q) = (q - q_0) \sum_{n\geqslant 0} (\overline{q_0} q)^n \tag{3}$$

on B. Now, since $qq_0 = q_0 q$ on $L_{I_{q_0}}$, then γ coincides, on $L_{I_{q_0}}$, with the regular map $\Gamma : B \to \mathbb{H}$ defined by the power series

$$\Gamma(q) = \sum_{n\geqslant 0} q^{n+1} \overline{q_0}^n - \sum_{n\geqslant 0} q^n \overline{q_0}^n q_0. \tag{4}$$

Therefore by Corollary 3.2, if γ were regular, then $\gamma \equiv \Gamma$ on the entire open set B. This fact would force the two power series (3) and (4) to be the same. This is not the case, and hence γ is not regular on B. □

Nevertheless, the following proposition holds true:

Proposition 4.3. *For any $u \in \mathbb{H}$, with $|u| = 1$, and any $q_0 \in B$, the map η defined by*

$$\eta(q) = u(q - q_0)(1 - \overline{q_0}q)^{-1}$$

is a diffeomorphism of B (with respect to its real structure).

Proof. For any given $q \in B$, we have

$$\begin{aligned}
1 - \bigl|\eta(q)\bigr|^2 &= 1 - (q - q_0)(1 - \overline{q_0}q)^{-1}\overline{(1 - \overline{q_0}q)^{-1}}\ \overline{(q - q_0)} \\
&= 1 - (q - q_0)\bigl[\overline{(1 - \overline{q_0}q)}(1 - \overline{q_0}q)\bigr]^{-1}\ \overline{(q - q_0)} = 1 - \frac{(q - q_0)(\bar{q} - \overline{q_0})}{|1 - \overline{q_0}q|^2} \\
&= \frac{(1 - \overline{q_0}q)(1 - \bar{q}q_0) - (\bar{q} - \overline{q_0})(q - q_0)}{|1 - \overline{q_0}q|^2} \\
&= \frac{1 - \overline{q_0}q - \bar{q}q_0 + |q_0|^2|q|^2 - (|q|^2 + |q_0|^2 - \bar{q}q_0 - \overline{q_0}q)}{|1 - \overline{q_0}q|^2} \\
&= \frac{1 + |q_0|^2|q|^2 - |q|^2 - |q_0|^2}{|1 - \overline{q_0}q|^2} = \frac{1 - |q|^2 - |q_0|^2(1 - |q|^2)}{|1 - \overline{q_0}q|^2} \\
&= \frac{(1 - |q|^2)(1 - |q_0|^2)}{|1 - \overline{q_0}q|^2} > 0
\end{aligned}$$

that is $1 - |\eta(q)|^2 > 0$ which implies $\eta(B) \subseteq B$. We will now prove that the map $\delta : B \to B$ defined by

$$\delta(q) = \bar{u}(q + uq_0)(1 + \overline{q_0}\,\bar{u}q)^{-1}$$

is the inverse of η. In fact, for any $q \in B$ we have

$$\begin{aligned}
\delta\bigl(\eta(q)\bigr) &= \bar{u}\bigl[u(q - q_0)(1 - \overline{q_0}q)^{-1} + uq_0\bigr]\bigl[1 + \overline{q_0}\,\bar{u}u(q - q_0)(1 - \overline{q_0}q)^{-1}\bigr]^{-1} \\
&= \left[(q - q_0)\frac{(1 - \bar{q}q_0)}{|1 - \overline{q_0}q|^2} + q_0\right]\left[1 + \frac{\overline{q_0}(q - q_0)(1 - \bar{q}q_0)}{|1 - \overline{q_0}q|^2}\right]^{-1} \\
&= \left[\frac{(q - q_0)(1 - \bar{q}q_0) + q_0(1 - \overline{q_0}q)(1 - \bar{q}q_0)}{|1 - \overline{q_0}q|^2}\right] \\
&\quad \times \left[\frac{(1 - \overline{q_0}q)(1 - \bar{q}q_0) + \overline{q_0}(q - q_0 - |q|^2q_0 + q_0\bar{q}q_0)}{|1 - \overline{q_0}q|^2}\right]^{-1} \\
&= \left[\frac{q - q_0 - |q|^2q_0 + q_0\bar{q}q_0 + q_0(1 - \overline{q_0}q - \bar{q}q_0 + |q_0|^2|q|^2)}{|1 - \overline{q_0}q|^2}\right] \\
&\quad \times \left[\frac{(1 - \bar{q}q_0 - |q_0|^2 + |q_0|^2\bar{q}q_0)}{|1 - \overline{q_0}q|^2}\right]^{-1}
\end{aligned}$$

$$
\begin{aligned}
&= \left[\frac{q - q_0 - |q|^2 q_0 + q_0\bar{q}q_0 + q_0 - |q_0|^2 q - q_0\bar{q}q_0 + q_0|q_0|^2|q|^2}{|1-\overline{q_0}q|^2}\right] \\
&\quad \times \left[\frac{(1-\bar{q}q_0 - |q_0|^2(1-\bar{q}q_0))}{|1-\overline{q_0}q|^2}\right]^{-1} \\
&= \left[\frac{q - |q|^2 q_0 - |q_0|^2 q + q_0|q_0|^2|q|^2}{|1-\overline{q_0}q|^2}\right]\left[\frac{(1-|q_0|^2)(1-\bar{q}q_0)}{|1-\overline{q_0}q|^2}\right]^{-1} \\
&= \left[\frac{q(1-|q_0|^2) - q_0|q|^2(1-|q_0|^2)}{|1-\overline{q_0}q|^2}\right]\left[\frac{(1-|q_0|^2)(1-\bar{q}q_0)}{|1-\overline{q_0}q|^2}\right]^{-1} \\
&= \left[\frac{(q - q_0|q|^2)(1-|q_0|^2)}{|1-\overline{q_0}q|^2}\right]\left[\frac{(1-|q_0|^2)(1-\bar{q}q_0)}{|1-\overline{q_0}q|^2}\right]^{-1} \\
&= \left[\left(q - q_0|q|^2\right)\right]\left[(1-\bar{q}q_0)\right]^{-1} = \frac{(q - q_0|q|^2)(1-\overline{q_0}q)}{|1-\overline{q_0}q|^2} \\
&= \frac{(q - q_0|q|^2 - q\overline{q_0}q + q|q_0|^2|q|^2)}{|1-\overline{q_0}q|^2} = \frac{q(1-\overline{q_0}q) - q(1-\overline{q_0}q)\bar{q}q_0}{|1-\overline{q_0}q|^2} \\
&= \frac{q(1-\overline{q_0}q)(1-\bar{q}q_0)}{|1-\overline{q_0}q|^2} = q.
\end{aligned}
$$

The proof of the relation $\eta(\delta(q)) = q$ is similar. □

The situation is further complicated by the fact that composition of regular functions is not regular in general, and therefore the set of all biregular transformations of the open unit ball B of $\mathbb{H}$ is not a group under composition. Still, something can be said by using a biregular Cayley-like transformation.

Define the quaternionic right-half space as $\mathbb{H}^+ = \{q = x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{H}\colon\ x_0 > 0\}$, and define the Cayley transform $\psi(q) = (1-q)^{-1}(1+q)$. Then we have the following results.

Lemma 4.4. *The Cayley transform is a regular function of q.*

Proof. Simply note that

$$\psi(q) = (1-q)^{-1}(1+q) = \left(\sum_{n\geqslant 0} q^n\right)(1+q)$$

is a series in q with real coefficients. □

Lemma 4.5. *The Cayley transform ψ maps B into $\mathbb{H}^+$.*

Proof. We need to show that if $q \in B$, then $\mathrm{Re}(\psi(q)) > 0$. This follows from the following steps:

$$\begin{aligned}
2\,\mathrm{Re}\big((1-q)^{-1}(1+q)\big) &= (1-q)^{-1}(1+q) + \overline{(1+q)}\,\overline{(1-q)^{-1}} \\
&= \frac{(1-\bar{q})}{|1-q|^2}(1+q) + (1+\bar{q})\frac{\overline{(1-\bar{q})}}{|1-q|^2} \\
&= \frac{(1-\bar{q})(1+q)}{|1-q|^2} + \frac{(1+\bar{q})(1-q)}{|1-q|^2} = 2\frac{(1-|q|^2)}{|1-q|^2} > 0. \qquad \square \qquad (5)
\end{aligned}$$

Lemma 4.6. *The function $\phi(w) = (w-1)(w+1)^{-1}$ is the regular inverse of the Cayley transform.*

Proof. The regularity of ϕ is immediately verified as in Lemma 4.4. To demonstrate that ψ and ϕ are one the inverse of the other, we simply need to show that $\phi(\psi(q)) = q$ and that $\psi(\phi(w)) = w$. Since the two verifications are similar, we will just do the first one. Indeed,

$$\begin{aligned}
\phi\big(\psi(q)\big) &= \big((1-q)^{-1}(1+q) - 1\big)\big((1-q)^{-1}(1+q) + 1\big)^{-1} \\
&= \big(2q(1-q)^{-1}\big)\big(2(1-q)^{-1}\big)^{-1} = q. \qquad \square
\end{aligned}$$

The combination of the lemmas above proves the following result.

Theorem 4.7. *The quaternionic right-half space $\mathbb{H}^+$ is diffeomorphic to the open unit ball B via the biregular Cayley transformation ψ.*

Proof. Immediate consequence of the previous lemmas. $\square$

5. Zeroes of quaternionic power series

At the beginning of this last section, we prove a result which extends to the case of power series an earlier result of Pogorui and Shapiro (see [11]). Their result was formulated for polynomials in the variable q, and its proof was rather elaborate. What we provide here generalizes the result to the case of power series, and incidentally offers a shorter proof of the result of [11].

Theorem 5.1. *Let $\sum_{n=0}^{+\infty} q^n a_n$ be a given quaternionic power series with radius of convergence R. Suppose that there exist $x_0, y_0 \in \mathbb{R}$ and $I, J \in \mathbb{S}$ with $I \neq J$ such that*

$$\sum_{n=0}^{+\infty} (x_0 + y_0 I)^n a_n = 0 \tag{6}$$

and

$$\sum_{n=0}^{+\infty} (x_0 + y_0 J)^n a_n = 0. \tag{7}$$

Then for all $L \in \mathbb{S}$ we have

$$\sum_{n=0}^{+\infty}(x_0 + y_0 L)^n a_n = 0.$$

Proof. For any fixed $n \in \mathbb{N}$ and any $L \in \mathbb{S}$ we have that

$$(x_0 + y_0 L)^n = \sum_{i=0}^{n} \binom{n}{i} x_0^{n-i} y_0^i L^i = \alpha_n + L\beta_n \tag{8}$$

where

$$\alpha_n = \left(\sum_{i \equiv 0 \pmod 4} \binom{n}{i} x_0^{n-i} y_0^i - \sum_{i \equiv 2 \pmod 4} \binom{n}{i} x_0^{n-i} y_0^i \right) \tag{9}$$

and

$$\beta_n = \left(\sum_{i \equiv 1 \pmod 4} \binom{n}{i} x_0^{n-i} y_0^i - \sum_{i \equiv 3 \pmod 4} \binom{n}{i} x_0^{n-i} y_0^i \right). \tag{10}$$

Equations (6), (7) and (8) yield, by absolute convergence,

$$\begin{aligned} 0 &= \sum_{n=0}^{+\infty}(\alpha_n + I\beta_n)a_n - \sum_{n=0}^{+\infty}(\alpha_n + J\beta_n)a_n \\ &= \sum_{n=0}^{+\infty}\big((\alpha_n + I\beta_n) - (\alpha_n + J\beta_n)\big)a_n \\ &= \sum_{n=0}^{+\infty}(I\beta_n - J\beta_n)a_n = (I - J)\left(\sum_{n=0}^{+\infty}\beta_n a_n\right). \end{aligned}$$

Therefore, being $I - J \neq 0$ by hypothesis, we get

$$\sum_{n=0}^{+\infty}\beta_n a_n = 0$$

and, by (6)

$$0 = \sum_{n=0}^{+\infty}(x_0 + y_0 I)^n a_n = \sum_{n=0}^{+\infty}(\alpha_n + I\beta_n)a_n = \sum_{n=0}^{+\infty}\alpha_n a_n + I\left(\sum_{n=0}^{+\infty}\beta_n a_n\right) = \sum_{n=0}^{+\infty}\alpha_n a_n.$$

Now, for any $L \in \mathbb{S}$ we have that

$$\sum_{n=0}^{+\infty}(x_0 + y_0 L)^n a_n = \sum_{n=0}^{+\infty}(\alpha_n + L\beta_n)a_n = \sum_{n=0}^{+\infty}\alpha_n a_n + L\left(\sum_{n=0}^{+\infty}\beta_n a_n\right) = 0$$

which proves the assertion. □

We now want to present a few properties which refine the splitting lemma 2.5, and which are of independent interest, as they help understand the geometry of these regular functions.

Proposition 5.2. *Let f be a regular function on a ball B centered in the origin. If there are two distinct imaginary units I and J in $\mathbb{S}$ such that $f(L_I) \subset L_I$ and $f(L_J) \subset L_J$, then f has a series representation*

$$f(q) = \sum q^n a_n$$

with real coefficients a_n.

Proof. If I and J are as in the hypothesis, we have that $f_I(x) = f_J(x)$ for any real number x. Therefore, if we write

$$f_I(x + yI) = f_0(x + yI) + f_1(x + yI)I$$

and

$$f_J(x + yJ) = g_0(x + yJ) + g_1(x + yJ)J,$$

we obtain that $f_0(x) + f_1(x)I = g_0(x) + g_1(x)J$ and therefore $f_0(x) = g_0(x)$, while $f_1(x) = g_1(x) = 0$. Consider now the functions $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ and $\beta : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\alpha(x, y) = (f_0(x + yI), f_1(x + yI))$ and $\beta(x, y) = (g_0(x + yI), g_1(x + yI))$. One immediately sees that both α and β satisfy the Cauchy–Riemann system of equations, and therefore they are holomorphic functions. However, since they coincide on $y = 0$, by the classical identity principle they coincide everywhere. This implies that $f_I = f_0(x + yI) + f_1(x + yI)I$ and that $f_J = f_0(x + yJ) + f_1(x + yJ)J$. We now expand f in power series, first using L_I and then using L_J. We obtain

$$f_I = \sum_{n \geqslant 0} (x + yI)^n \left(a_n^0 + a_n^1 I\right)$$

and

$$f_J = \sum_{n \geqslant 0} (x + yJ)^n \left(a_n^0 + a_n^1 J\right).$$

Since the two expressions must coincide when written for generic real numbers, we obtain that, for every n, it is

$$a_n^0 + a_n^1 I = a_n^0 + a_n^1 J.$$

This implies that $a_n^1 = 0$ for all n, and therefore the series representation of f has real coefficients. □

Theorem 5.1 points out a curious property of the zero set of regular functions, whose structure turns out to be significantly different from the one of the zero set of holomorphic functions. The zeros become even nicer in the following case:

Proposition 5.3. *If f has a series representation $f(q)=\sum q^n a_n$ with real coefficients a_n, then every real zero x_0 is isolated, and if $x_0+y_0I_0$ is a non-real zero* (*i.e.* $y_0\neq 0$) *then x_0+y_0I is a zero for any $I\in\mathbb{S}$. In particular, if $f\not\equiv 0$, the zero set of f consists of isolated points* (*belonging to* $\mathbb{R}$) *or isolated 2-spheres.*

Proof. As we established in (8), for any $n\in\mathbb{N}$ we have

$$(x_0+y_0I_0)^n=\sum_{i=0}^{n}\binom{n}{i}x_0^{n-i}y_0^iI_0^i=\alpha_n+I_0\beta_n$$

where α_n and β_n are the real numbers defined in (9) and (10). Now, if $x_0+y_0I_0$ is a non-real zero of f we obtain

$$0=\sum_{n=0}^{+\infty}(x_0+y_0I_0)^n a_n=\sum_{n=0}^{+\infty}(\alpha_n+I_0\beta_n)a_n=\sum_{n=0}^{+\infty}\alpha_n a_n+I_0\left(\sum_{n=0}^{+\infty}\beta_n a_n\right)$$

which, being a_n real for all n by hypothesis, implies

$$\sum_{n=0}^{+\infty}\alpha_n a_n=0=\left(\sum_{n=0}^{+\infty}\beta_n a_n\right)$$

and hence, for any $I\in\mathbb{S}$,

$$0=\sum_{n=0}^{+\infty}\alpha_n a_n+I\left(\sum_{n=0}^{+\infty}\beta_n a_n\right)=\sum_{n=0}^{+\infty}(x_0+y_0I)^n a_n$$

i.e. x_0+y_0I is a zero of f for all $I\in\mathbb{S}$. To conclude the proof, we observe that for any I in $\mathbb{S}$ the plane L_I contains all the real zeros of f and exactly two zeros for each sphere of zeros of f. Since the zeros of f have to be isolated points on L_I unless $f\equiv 0$ (see Theorem 3.1), the zero set of f consists of isolated (real) points and isolated 2-spheres. □

If a regular function f is the extension of a holomorphic function of L_I, for some $I\in\mathbb{S}$, then its zeroes have very nice properties as well. Namely

Proposition 5.4. *Let f be a regular function on a ball B centered in the origin, and suppose that there exists an imaginary unit $I\in\mathbb{S}$ such that $f(L_I)\subset L_I$. If there exists an imaginary unit $J\in\mathbb{S}$ such that $J\notin L_I$ and that $f(x_0+y_0J)=0$, then $f(x_0+y_0L)=0$ for all $L\in\mathbb{S}$. In particular, if $f\not\equiv 0$, the zero set of f consists of isolated points of B* (*belonging to* L_I) *or isolated 2-spheres of B.*

Proof. By Corollary 2.8, the inclusion $f(L_I)\subset L_I$, implies that f has a power series representation

$$f(q)=\sum q^n a_n$$

with coefficients $a_n \in L_I$, for all $n \in \mathbb{N}$. By assumption and by (8) we have

$$0 = \sum_{n=0}^{+\infty} (x_0 + y_0 J)^n a_n = \sum_{n=0}^{+\infty} (\alpha_n + J\beta_n) a_n = \sum_{n=0}^{+\infty} \alpha_n a_n + J\left(\sum_{n=0}^{+\infty} \beta_n a_n\right) \tag{11}$$

with $\alpha_n, \beta_n \in \mathbb{R}$ for all $n \in \mathbb{N}$. If we now set

$$A = \sum_{n=0}^{+\infty} \alpha_n a_n, \qquad B = \sum_{n=0}^{+\infty} \beta_n a_n$$

we obtain $A + JB = 0$ with A and $B \in L_I$. Therefore $A = B = 0$ and hence obviously $A + LB = 0$ for all $L \in \mathbb{S}$. The last part of the assertion follows from the properties of the zero-set of holomorphic functions. $\square$

This result has a curious consequence concerning the zeroes of holomorphic functions. Since (the power series expansion of) any holomorphic function f can be uniquely extended to (the power series expansion of) a regular function over quaternions, the question of distinguishing which zeroes of f will remain isolated after the extension, and which will become "spherical," naturally arises. For $I \in \mathbb{S}$, let $\Delta_I(0, R) = \{x + yI\colon\ x^2 + y^2 < R^2\}$, and let $f : \Delta_I(0, R) \to \mathbb{R} + \mathbb{R}I$ be holomorphic. If, for $\{r_n\}, \{s_n\} \subset \mathbb{R}$,

$$f(x + yI) = \sum_{n=0}^{+\infty} (x_0 + y_0 I)^n (r_n + s_n I)$$

is the power series expansion of f, then $f(x + yI) = 0$ can be written in terms of (8) as

$$\left(\sum_{n=0}^{+\infty} \alpha_n (r_n + s_n I)\right) + I\left(\sum_{n=0}^{+\infty} \beta_n (r_n + s_n I)\right) = 0 \tag{12}$$

and as

$$\left(\sum_{n=0}^{+\infty} \alpha_n r_n - \sum_{n=0}^{+\infty} \beta_n s_n\right) + I\left(\sum_{n=0}^{+\infty} \alpha_n s_n + \sum_{n=0}^{+\infty} \beta_n r_n\right) = 0$$

i.e.

$$\left(\sum_{n=0}^{+\infty} \alpha_n r_n - \sum_{n=0}^{+\infty} \beta_n s_n\right) = 0 = \left(\sum_{n=0}^{+\infty} \alpha_n s_n + \sum_{n=0}^{+\infty} \beta_n r_n\right).$$

Proposition 5.5. *Let $f : \Delta_I(0, R) \to \mathbb{R} + \mathbb{R}I$ be holomorphic, let $\tilde{f} : B(0, R) \to \mathbb{H}$ be the regular extension of f, and let $\tilde{f}(x_0 + y_0 I) = 0$. The zero $(x_0 + y_0 I)$ of $\tilde{f}$ is not isolated, or equivalently $\tilde{f}(x_0 + y_0 L) = 0$ for all $L \in \mathbb{S}$, if, and only if,*

$$0=\sum_{n=0}^{+\infty}\alpha_n r_n=\sum_{n=0}^{+\infty}\beta_n s_n=\sum_{n=0}^{+\infty}\alpha_n s_n=\sum_{n=0}^{+\infty}\beta_n r_n.$$

Proof. The fact that $\tilde{f}(x_0+y_0L)=0$ for all $L\in\mathbb{S}$ is, by (11), equivalent to

$$\left(\sum_{n=0}^{+\infty}\beta_n(r_n+s_nI)\right)=0.$$

The assertion follows immediately from (12). □

A more refined result is the following:

Proposition 5.6. *Let $f\not\equiv 0$, and let $I\in\mathbb{S}$ be such that $f(L_I)\subset L_I$. Then, for any $J\neq I$, $J\in\mathbb{S}$, we have that $f(L_J)\not\subset(L_I)^\perp$.*

Proof. First we note that, by Corollary 2.8, the condition $f(L_I)\subset L_I$ implies that

$$f_I(x+yI)=\sum_{n\geqslant 0}(x+yI)^n\left(a_n^0+a_n^1I\right).$$

Take now a different imaginary unit J and suppose that

$$f(L_J)\subset(L_I)^\perp.$$

Then we obtain

$$f_J(x+yJ)=\sum_{n\geqslant 0}(x+yJ)^n\left(b_n^0\tilde{J}+b_n^1\tilde{K}\right)$$

where $Span(\tilde{J},\tilde{K})\perp L_I$. Therefore $a_n^0+a_n^1I=b_n^0\tilde{J}+b_n^1\tilde{K}$ and so $a_n^0=a_n^1=b_n^0=b_n^1=0$ and $f\equiv 0$ which proves the proposition. □

In addition we show that

Proposition 5.7. *Let $f\not\equiv 0$, and let $I_0\in\mathbb{S}$ be such that $f(L_{I_0})\not\subset L_{I_0}$. Let $J\in\mathbb{S}$ be any imaginary unit orthogonal to I_0, and let F_0 and G_0 be two holomorphic functions such that $f_{I_0}=F_0+G_0J$. Then*:

1. *$G_0\not\equiv 0$.*
2. *If there exists $q\in L_{I_0}$ such that $f(q)=0$, then neither F_0 nor G_0 are identically constant* (*except that F_0 may be identically zero*).
3. *If there exists an imaginary unit $T\not\perp I_0$ such that $f(L_T)\subset L_T$, then $F_0\not\equiv 0$.*

Proof. The proof of (1) and (2) is straightforward. To prove (3) we proceed by contradiction. If $F_0\equiv 0$, then

$$f_{I_0}(x+yI_0)=\sum_{n\geqslant 0}(x+yI_0)^n\left(a_n^0J+a_n^1K\right)$$

where $K=I_0J$. Moreover, since $f(L_T)\subseteq L_T$ we get

$$f_T(x+yT)=\sum_{n\geqslant 0}(x+yT)^n\left(b_n^0+b_n^1T\right).$$

Now, the uniqueness of the quaternionic series expansion of f yields

$$\left(a_n^0J+a_n^1K\right)=\left(b_n^0+b_n^1T\right).$$

The hypothesis $T\not\perp I_0$ implies that $T\notin\mathbb{R}J+\mathbb{R}K$ and hence that $a_n^0=a_n^1=b_n^0=b_n^1=0$ for all n, contradicting the hypothesis $f\not\equiv 0$. □

Finally, we conclude with a simple and nice result which we had originally announced in [6]

Proposition 5.8. *Let f be a regular function on B with $f\not\equiv 0$. For any imaginary unit $I\in\mathbb{S}$, let $J_I=J$ be an imaginary unit orthogonal to I, and let $F_I, G_I: L_I\cap B\to L_I$ be the holomorphic functions such that $f_I=F_I+G_IJ$. Then*:

1. *Either $G_I\equiv 0$ on L_I for every I, or there is at most one imaginary unit I such that $G_I\equiv 0$ on L_I.*
2. *If $F_{I_1}\equiv 0$ on L_{I_1} and $F_{I_2}\equiv 0$ on L_{I_2} for $I_1\neq I_2$, then $F_T\equiv 0$ on L_T for all imaginary units $T\in\mathbb{R}I_1+\mathbb{R}I_2$.*
3. *If there exists an imaginary unit I such that $G_I\equiv 0$ on L_I (respectively $F_I\equiv 0$ on L_I), then there is no other unit $I'\not\perp I$ such that $F_{I'}\equiv 0$ on $L_{I'}$ (respectively $G_{I'}\equiv 0$ on $L_{I'}$).*

Proof. (1) If there exist two imaginary units $I_1\neq I_2$ such that $G_{I_1}\equiv 0$ on L_{I_1} and $G_{I_2}\equiv 0$ on L_{I_2}, then $F(L_{I_1})\subseteq L_{I_1}$ and $F(L_{I_2})\subseteq L_{I_2}$. Therefore (see Corollary 2.8) we obtain

$$f_{I_1}(x+yI_1)=\sum_{n\geqslant 0}(x+yI_1)^n\left(a_n^0+a_n^1I_1\right)\quad\text{on } L_{I_1}\cap B$$

and

$$f_{I_2}(x+yI_2)=\sum_{n\geqslant 0}(x+yI_2)^n\left(b_n^0+b_n^1I_2\right)\quad\text{on } L_{I_2}\cap B.$$

The uniqueness of the power series expansion of f yields $(a_n^0+a_n^1I_1)=(b_n^0+b_n^1I_2)$, and hence $a_n^0=b_n^0$, $a_n^1=b_n^1=0$, for all n. Therefore, for any $I\in\mathbb{S}$ the series

$$f_I(x+yI)=\sum_{n\geqslant 0}(x+yI)^na_n^0$$

has values in L_I, that is $G_I\equiv 0$ on L_I.

(2) If $F_{I_1} \equiv 0$ on L_{I_1} and $F_{I_2} \equiv 0$ on L_{I_2} then, for $J \in \mathbb{S}$ perpendicular both to I_1 and to I_2, we have

$$f_{I_1}(x + yI_1) = \sum_{n \geqslant 0} (x + yI_1)^n \left(a_n^0 + a_n^1 I_1\right) J \quad \text{on } L_{I_1} \cap B$$

and

$$f_{I_2}(x + yI_2) = \sum_{n \geqslant 0} (x + yI_2)^n \left(b_n^0 + b_n^1 I_2\right) J \quad \text{on } L_{I_2} \cap B.$$

The uniqueness of the power series expansion of f implies that $(a_n^0 + a_n^1 I_1)J = (b_n^0 + b_n^1 I_2)J$, and hence $a_n^0 = b_n^0$, $a_n^1 = b_n^1 = 0$, for all n. Now, for any imaginary unit $T \in \mathbb{R}I_1 + \mathbb{R}I_2 \subseteq (J)^\perp$ the series

$$f_T(x + yT) = \sum_{n \geqslant 0} (x + yT)^n a_n^0 J$$

has values in $\mathbb{R}J + \mathbb{R}(TJ) = (L_T)^\perp$, i.e. $F_T \equiv 0$ on L_T.

(3) If there are imaginary units $I \neq I'$ such that $G_I \equiv 0$ on L_I and $F_{I'} \equiv 0$ on $L_{I'}$, then with $J \in \mathbb{S}$ as before perpendicular both to I and to I', we get (see Corollary 2.8)

$$f_I(x + yI) = \sum_{n \geqslant 0} (x + yI)^n \left(a_n^0 + a_n^1 I\right) \quad \text{on } L_I \cap B$$

and

$$f_{I'}(x + yI') = \sum_{n \geqslant 0} (x + yI')^n \left(b_n^0 + b_n^1 I'\right) J \quad \text{on } L_{I'} \cap B.$$

Since $I' \not\perp I$, then $I'J \notin \mathbb{R}I + \mathbb{R}J$, which again by the uniqueness of the series expansion of f yields $a_n^0 = b_n^0 = a_n^1 = b_n^1 = 0$ concluding the proof. □

Acknowledgments

The authors acknowledge the support of George Mason University during the preparation of this paper. This work has been partially supported by G.N.S.A.G.A. of INdAM and by MIUR.

References

[1] L. Ahlfors, Complex Analysis, McGraw–Hill, New York, 1966.
[2] F. Colombo, I. Sabadini, F. Sommen, D.C. Struppa, Analysis of Dirac Systems and Computational Algebra, Birkhäuser, 2004.
[3] C.G. Cullen, An integral theorem for analytic intrinsic functions on quaternions, Duke Math. J. 32 (1965) 139–148.
[4] R. Fueter, Die Funktionentheorie der Differentialgleichungen $\Delta u = 0$ und $\Delta\Delta u = 0$ mit vier reellen Variablen, Comment. Math. Helv. 7 (1934) 307–330.
[5] R. Fueter, Über Hartogs'schen Satz, Comment. Math. Helv. 12 (1939) 75–80.

[6] G. Gentili, D.C. Struppa, A new approach to Cullen-regular functions of a quaternionic variable, C. R. Acad. Sci. Paris Ser. I 342 (2006) 741–744.
[7] V.V. Kravchenko, M.V. Shapiro, Integral Representations for Spatial Models of Mathematical Physics, Pitman Res. Notes Math., vol. 351, Longman, Harlow, 1996.
[8] G. Laville, I. Ramadanoff, Holomorphic Cliffordian functions, Adv. Appl. Clifford Algebras 8 (1998) 323–340.
[9] G. Laville, I. Ramadanoff, Elliptic Cliffordian functions, Complex Var. Theory Appl. 45 (2001) 297–318.
[10] H. Leutwiler, Modified quaternionic analysis in $\mathbb{R}^3$, Complex Var. Theory Appl. 20 (1992) 19–51.
[11] A. Pogorui, M.V. Shapiro, On the structure of the set of zeros of quaternionic polynomials, Complex Variables 49 (6) (2004) 379–389.
[12] R.F. Rinehart, Elements of a theory of intrinsic functions on algebras, Duke Math. J. 27 (1960) 1–19.
[13] A. Sudbery, Quaternionic analysis, Math. Proc. Cambridge Philos. Soc. 85 (1979) 199–225.